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HYDRODYNAMIC INTERACTION OF TWO IDENTICAL LIQUID SPHERES IN LINEAR FLOW FIELD*

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In quasi-stationarly Stokes equations, the hydrodynamic interaction of two identical free of external forces fluid spherical particles is considered in linear flow field of arbitrary type. Numerical results are given about the particles relative velocity and intensity of force dipoles on spheres. Close-by and remote asymptotic expansions of respective hydrodynamic functions are obtained. The results are of interest in the structural rheology of emulsions.

Earlier, the hydrodynamic interaction of two solid spheres was investigated in linear flow field /1— 4/, as well as that of two drops in a medium which away from the particles is at rest. The used here investigation methods are similar to those defined in /5— 8/.

1. Statement of the problem. Consider two free of external forces liquid spherical particles of radius a and dynamical viscosity μ' submerged in an incompressible fluid of viscosity μ_e. The application of quasi-stationary Stokes equations is assumed inside the drops and in the external medium. As the boundary condition on the surface of spheres, we take the impenetrability of the liquid, and continuity of tangential stresses. The surface tension at the interface is assumed fairly large, which permits the neglect of particle deviation from spherical and not consider the boundary condition for normal stresses. The surrounding flow field is of the form

$$\mathbf{v}_\infty = \mathbf{v}_0 + \mathbf{\Omega} \times \mathbf{x} + \mathbf{E}\cdot\mathbf{x} \quad (\mathbf{v}_0, \mathbf{\Omega}, \mathbf{E} = \text{const})$$

where $\mathbf{\Omega}$, $\mathbf{E}$ are, respectively, the vorticity and the tensor of deformation rate of the unperturbed flow; and vector $\mathbf{x}$ is drawn from the coordinate origin (see Fig.1). The velocities of transfer of spheres are determined by the condition of hydrodynamic forces being zero. The equality of bieng zero of the moments of forces relative to the center of particles is satisfied automatically /7/.

According to the method /9/ for the application of this problem to the calculation of the mean stress in a monodisperse emulsion it is necessary to determine the velocity $\mathbf{V}$ of motion of sphere 1 relative to sphere 2, and, also, of the dipole force on sphere 1

$$S_{jk} = \int_{S_1} \left\{(\sigma_n)_j x_k - \frac{1}{3}\delta_{jk}(\sigma_n\cdot\mathbf{x}) - \mu_e(v_j n_k + v_k n_j)\right\} dS \tag{1.1}$$

where S_1, $\mathbf{n}$ is the sphere surface 1 and the external normal to it; $\mathbf{v}$ is the velocity of fluid; and vector of stresses σ_n is calculated on the outside of the surface /9/. Under conditions of the sphere free motion, the following general representations are valid /9/:

$$\begin{aligned} \mathbf{V} &= l\,[\mathbf{\Omega}\times\mathbf{p} + (1-B)\,\mathbf{E}\cdot\mathbf{p} + (B-A)\,(\mathbf{p}\cdot\mathbf{E}\cdot\mathbf{p})\mathbf{p}] \\ \mathbf{S} &= {}^4/_3\pi a^3\,(2+5\lambda)(1+\lambda)^{-1}\mu_e\,\{(1+K)\,\mathbf{E} + [(\mathbf{E}\cdot\mathbf{p})\,\mathbf{p} + \\ &\quad \mathbf{p}\,(\mathbf{E}\cdot\mathbf{p})]L + (\mathbf{p}\cdot\mathbf{E}\cdot\mathbf{p})[\mathbf{pp}M - ({}^2/_3 L + {}^1/_3 M)\mathbf{I}]\},\ \lambda = \mu'/\mu_e \end{aligned} \tag{1.2}$$

where l is the distance between the centers of spheres, $\mathbf{p}$ is the unit vector of the line of centers, directed from sphere 2 to sphere 1, $\mathbf{I}$ is the unit tensor. The basic aim of this work is the calculation and investigation of scalar functions A, B, K, L, M defining the interaction between spheres dependent on λ, ε (εa is the clearance between spheres). It suffices to consider three separate problems (the system of coordinates x_1, x_2, x_3 is introduced, as shown in Fig.1).

Problem 1. $\mathbf{v}_\infty = (E_{11}x_1, -E_{11}x_2, 0)$. The stream does not act on the spheres. The solution yields the value of K.

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Problem 2. $\mathbf{v}_\infty = (Gx_3, 0, 0)$. The shift field gives rise to sphere velocity $(\pm V_1, 0, 0)$. The solution yields values of B and $K + L$.

Problem 3. $\mathbf{v}_\infty = (-{}^1/_2E_{33}x_1, -{}^1/_2E_{33}x_2, E_{33}x_3)$. Under the action of this deformation axisymmetric flow, the spheres acquire the velocities $(0, 0, \pm V_1^*)$. The solution provides the possibility to calculate A and $K + {}^4/_3L + {}^2/_3M$.

Note that the used here methods of investigation of problems 1 and 3 are simpler than the application of general representation /1/ of solution of Stokes equations in bispheric coordinates.

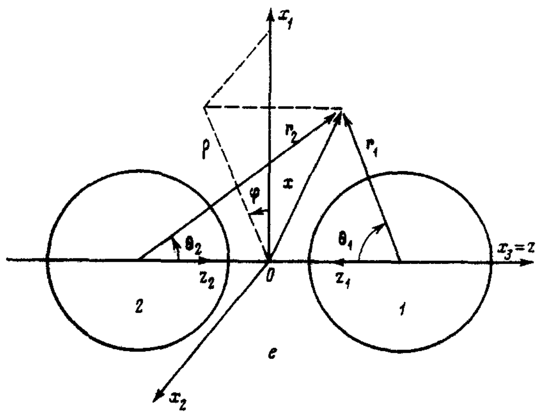

Fig.1

2. Solution of problem 1.

We apply the method of multiple reflections, similarly to /8/, which is equivalent to the construction of the exact solution. We pass to dimensionless variables normalizing the velocities with respect to $E_{11}l$, and lengths to l. We seek the velocity field in region e (between spheres) in the form

$$\mathbf{v}^e = \mathbf{v}_\infty + \mathbf{v}^* + \mathbf{v}^{**}, \quad \mathbf{v}^*, \mathbf{v}^{**} \to 0 \quad (|\mathbf{x}| \to \infty)$$

The boundary conditions for $\mathbf{v}^*$ are as follows: the field $\mathbf{v}_\infty + \mathbf{v}^*$ $(\mathbf{v}^*)$ has zero normal component on the sphere 1 (2); inside the sphere 1 (2) we have a Stokes flow which has at the boundary the same velocity and tangential stress as the field $\mathbf{v}_\infty + \mathbf{v}^*$ $(\mathbf{v}^*)$. To formulate boundary conditions for $\mathbf{v}^{**}$ we change the places of spheres. It is sufficient to consider field $\mathbf{v}^*$ which we seek by the method of reflection in the form

$$\mathbf{v}^* = \sum_{k=1}^{\infty} (\mathbf{v}_-^{1,2k-1} + \mathbf{v}_-^{2,2k}) \tag{2.1}$$

Each field $\mathbf{v}_-^{i,j}$ satisfies the Stokes equations and is regular everywhere outside the i-the sphere and vanishes at infinity.

The calculations are carried out in succession

$$\mathbf{v}_+^{i,j} \to \mathbf{v}_-^{i,j+1} \to \mathbf{v}_+^{i+1,j+1} \to \mathbf{v}_-^{i+1,j+2} \to \mathbf{v}_+^{i,j+2} \to \ldots \tag{2.2}$$

where the initial $\mathbf{v}_+^{1,0} = \mathbf{v}_\infty$, and $\mathbf{v}_+^{i+1,j}$ $(j \geqslant 1)$ denotes the expansion of field $\mathbf{v}_-^{i,j}$ in the neighborhood $(i+1)$-st sphere (indices i, $i+1$ are deduced in modulo 2). The transition from $\mathbf{v}_+^{i,j}$ to $\mathbf{v}_-^{i,j+1}$ is determined by the boundary conditions, as in /8/.

We introduce two spherical systems of coordinates $(r_1, \theta_1, \varphi_1)$, $(r_2, \theta_2, \varphi_2)$, as shown in Fig.1. Angle φ_i corresponds to positive direction of rotation around the axis z_i $(\varphi_2 = -\varphi_1 = \varphi)$. Using the Lamb general solution of Stokes equations, we represent the sought field in the form

$$\mathbf{v}_\pm^{i,j} = \sum_{n=2}^{\infty} \left[\operatorname{rot}(\mathbf{r}_i\chi_k^{i,j}) + \nabla\Phi_k^{i,j} + \frac{(k+3)\nabla(r_i^2 p_k^{i,j})}{2(k+1)(2k+3)} - \frac{\mathbf{r}_i p_k^{i,j}}{k+1}\right] \tag{2.3}$$

$$p_k^{i,j} = \zeta A_k^{i,j}\cos 2\varphi_i, \quad \Phi_k^{i,j} = \zeta B_k^{i,j}\cos 2\varphi_i$$

$$\chi_k^{i,j} = \zeta C_k^{i,j}\sin 2\varphi_i, \quad \zeta = r_i^k P_n^2(\cos\theta_i)$$

where $k = n$ for field $\mathbf{v}_+^{i,j}$ and $k = -(n+1)$ for $\mathbf{v}_-^{i,j}$; P_n^2 is the adjoint Legendre function. The selection of special form of spherical harmonics is dictated by the form of $\mathbf{v}_\infty$.

As in /8/, the formulas of transition from $\mathbf{v}_+^{i,j}$ to $\mathbf{v}_-^{i,j+1}$ are of the form

$$C_{-(n+1)}^{i,j+1} = \frac{(n-1)(1-\lambda)}{n+2+\lambda(n-1)} C_n^{i,j}\alpha^{2n+1}, \quad \alpha = \frac{a}{l} \tag{2.4}$$

$$A_{-(n+1)}^{i,j+1} = -\frac{n(2n-1)}{(n+1)(1+\lambda)}\left\{\frac{\lambda}{2} A_n^{i,j}\alpha^{2n+1} + [2+\lambda(2n+1)] B_n^{i,j}\alpha^{2n-1}\right\}$$

$$B_{-(n+1)}^{i,j+1} = \frac{n}{2(n+1)(1+\lambda)}\left\{\frac{2-\lambda(2n+1)}{2(2n+3)} A_n^{i,j}\alpha^{2n+3} - \lambda(2n-1) B_n^{i,j}\alpha^{2n+1}\right\}$$

For representing fields $\mathbf{v}_-^{i,j}$ in the form $\mathbf{v}_+^{i+1,j}$ in the neighborhood of $(i+1)$ sphere we have the relations

$$A_n^{i+1,j} = \sum_{m=2}^{\infty} g_n^m A_{-(m+1)}^{i,j}, \quad g_n^m = \frac{(n+m)!}{(m-2)!(n+2)!} \tag{2.5}$$

$$C_n^{i+1,j} = \sum_{m=2}^{\infty}\left[\frac{2}{mn(n+1)} g_n^m A_{-(m+1)}^{i,j} + \frac{m}{n+1} g_n^m C_{-(m+1)}^{i,j}\right]$$

$$B_n^{i+1,j} = \sum_{m=2}^{\infty}\left\{\frac{A_{-(m+1)}^{i,j}}{m(2m-1)}\left\{\frac{(n-2)[(m-2)(n-1)-(m+1)]}{n(2n-1)} g_{n-1}^m - \frac{m-2}{2} g_n^m\right\} + g_n^m B_{-(m+1)}^{i,j} + \frac{2}{n} g_n^m C_{-(m+1)}^{i,j}\right\}$$

The initial condition $\mathbf{v}_+^{1,0} = \mathbf{v}_\infty$ yields

$$A_n^{1,0} = C_n^{1,0} = 0 \quad (n \geqslant 2), \quad B_2^{1,0} = 1/_6, \quad B_n^{1,0} = 0 \quad (n \geqslant 3) \tag{2.6}$$

Using (2.1)– (2.6) it is possible to determine $\mathbf{v}^*$ and, by changing the places of spheres, also, $\mathbf{v}^{**}$. For the resulting field $\mathbf{v}^e$ the stress vector on sphere 1 is determined using general formulas /10/. Calculating integral (1.1), we find

$$K = \frac{1}{A_{-3}^{1,1}} \sum_{k=1}^{\infty} (A_{-3}^{1,2k+1} + A_{-3}^{2,2k}) \tag{2.7}$$

Analytic calculation of the first reflections using the scheme (2.2)– (2.7) yields the distant asymptotic representation

$$K = -\frac{2\lambda}{1+\lambda}\alpha^5 + \frac{2+5\lambda}{3(1+\lambda)}\left[\frac{\lambda-1}{\lambda+4} + \frac{2+7\lambda}{8(1+\lambda)}\right]\alpha^8 + O(\alpha^{10}), \quad \alpha \to 0 \tag{2.8}$$

Ths subsequent reflections give

$$K = \sum_{n=5}^{\infty} c_n (2\alpha)^n \quad (c_6 = c_7 = c_9 = c_{11} = 0) \tag{2.9}$$

The coefficients $c_n(\lambda)$ at $n \leqslant 172$ were calculated using scheme (2.2)– (2.7) on a computer, similarly to /8/.

Computation has shown a satisfactory rate of convergence of series (2.9) for any λ even for touching particles. For example, for $\lambda = \infty$ and $\varepsilon = 0$ the calculated with the use of (2.9) for $n \leqslant 76$ and $n \leqslant 172$ the values of K were, respectively, $-0{,}04717$ and $-0{,}04721$, and were to the third digit the same as those /4/ obtained by exact solution in tangentially spherical coordinates.

The values of $-K \times 10^4$ are shown in Table 1. For $\varepsilon \geqslant 1$ and $\lambda \geqslant 0.5$ the relative error of formula (2.8) does not exceed 0.6%.

Table 1

λ	ε=1	0.25	0.1	0.01	0
0.5	27	113	161	204	209
1	40	166	234	292	299
2	53	216	300	368	377
5	66	259	354	429	439
10	71	275	373	450	459
∞	77	290	389	463	472

For $\lambda = 0$ it is possible using (2.4) and (2.5) to obtain

$$\gamma_n^{i+1,j} = (n-1)(n+2)\sum_{m=3}^{\infty}\frac{g_n^{m-1}\alpha^{2m-1}}{(n+m-1)(m-1)}\gamma_m^{i,j-1}, \tag{2.10}$$

$$n \geqslant 2, j \geqslant 1$$

$$\gamma_n^{i,j} = (n-1)(n+2)B_n^{i,j} + \frac{(n-2)(n-3)}{2(2n-1)}A_{n-2}^{i,j} + (n-1)(n-2)C_{n-1}^{i,j}$$

From (2.4), (2.6), (2.7) and (2.10) follows the unexpected vanishing of all coefficients c_n and function K when $\lambda = 0$.

3. Solution of problem 2. For the construction of exact solution with some simplifications we apply the method of /7/. We introduce the bispherical coordinates ξ, η, linked with the cylindrical coordinates ρ, z by the relations

$$z = \frac{c\,\mathrm{sh}\,\eta}{\mathrm{ch}\,\eta - \mu}, \quad \rho = \frac{c\sin\xi}{\mathrm{ch}\,\eta - \mu}, \quad \mu = \cos\xi \tag{3.1}$$

The spheres 1 and 2 are coordinate surfaces $\eta = \eta_1 > 0$ and $\eta = -\eta_1$, respectively, if we set $\mathrm{ch}\,\eta_1 = 1 + \varepsilon/2$, $c = a\,\mathrm{sh}\,\eta_1$. Satisfying Stokes equation $\mathrm{rot}\,(\Delta\mathbf{v}) = 0$, we seek the cylindrical velocity components in the region between spheres and inside them of the form

$$\begin{aligned} &v_\rho = V_1(\rho F/c + \chi + \psi)\cos\varphi \\ &v_\varphi = V_1(\chi - \psi)\sin\varphi, \quad v_z = V_1(zF/c + 2\Phi)\cos\varphi \\ &F = \zeta\sum_{n=1}^{\infty} f_n(\eta)P_n^1(\mu), \qquad \Phi = \zeta\sum_{n=1}^{\infty}\varphi_n(\eta)P_n^1(\mu) \\ &\chi = \zeta\sum_{n=2}^{\infty}\chi_n(\eta)P_n^2(\mu), \qquad \psi = \zeta\sum_{n=0}^{\infty}\psi_n(\eta)P_n(\mu), \qquad \zeta = (\mathrm{ch}\,\eta - \mu)^{1/2} \end{aligned} \tag{3.2}$$

where $P_n^m(\mu)$, $P_n(\mu)$ are, respectively the Legendre adjoint function and polynomial, and f_n, φ_n, χ_n, ψ_n are linear combination of functions $\exp[(n + 1/2)\eta]$, $\exp[-(n + 1/2)\eta]$. Using the transformation

$$\begin{aligned} &3(2n+1)f_n = (2n+1)\alpha_n + 2(n+2)\beta_n + 2(n-1)\gamma_n \\ &6(2n+1)\chi_n = -(2n+1)\alpha_n - (2n+7)\beta_n - (2n-5)\gamma_n \\ &6(2n+1)\psi_n = -(2n+1)n(n+1)\alpha_n - (n+1)(n+2)\cdot \\ &\quad (2n+3)\beta_n - n(n-1)(2n-1)\gamma_n \end{aligned} \tag{3.3}$$

we introduce the additional functions $\alpha_n, \beta_n, \gamma_n$.

The difference form the case in /7/ is stipulated by the symmetry of the problem and the inhomogeneity of the surrounding flow. We set

$$\begin{aligned} &\alpha_n^e = 2J_n^e\,\mathrm{sh}\,(n + 1/2)\eta + 2\psi_n^\infty \\ &(\beta_n^e, \gamma_n^e) = 2(L_n^e, N_n^e)\,\mathrm{sh}\,(n + 1/2)\eta - \psi_n^\infty \\ &\varphi_n^e = 2B_n^e\,\mathrm{ch}\,(n + 1/2)\eta, \; f_n^e = C_n^e\,\mathrm{sh}\,(n + 1/2)\eta \\ &(\alpha_n^i, \beta_n^i, \gamma_n^i, \varphi_n^i) = (\pm J_n, \pm L_n, \pm N_n, B_n)\exp[-(n + 1/2)|\eta|] \\ &\psi_n^\infty = \sqrt{2}(Gc/V_1)(2n+1)\exp[-(n + 1/2)|\eta|]\,\mathrm{sign}\,\eta \end{aligned} \tag{3.4}$$

The indexes $e, i\ (i = 1, 2)$ denote magnitudes related to regions shown in Fig.1; the upper sign corresponds to $i = 1$, the lower to $i = 2$. The inhomogeneous terms in expressions for $\alpha_n^e, \beta_n^e, \gamma_n^e$, after substitutions into (3.3) and, then, into (3.2), yield the unperturbed flow $\mathbf{v}_\infty$; the perturbation of velocity, determined by the additional terms, vanish at infinity. Now for the construction of difference equations, defining the eight independent sequencies $J_n^e, J_n, \ldots, B_n^e, B_n$, the results of /7/ are directly applicable. The solenoidality of flow in all regions with allowance for the continuity of velocity, and also the boundary conditions of impermeability constitute a system of three linear equations for J_n^e, L_{n-1}^e, N_{n+1}^e (the second of relations (1.8), the second of equalities (1.10) and relation (1.12) for $i = 1, \delta_1 = 1$ of /7/) from which we obtain the formulas for the unknowns in terms of $B_m^e, B_m\ (n - 1 \leqslant m \leqslant n + 1)$. As the result, formulas (3.3) yield the representation of $f_n^e, \chi_n^e, \psi_n^e$ in terms of B_m^e, $B_m\ (n - 2 \leqslant m \leqslant n + 2)$; a similar representation of $f_n^i, \chi_n^i, \psi_n^i$ in terms of basic variables B_m^e and B_m are obtained form the conditions of velocity continuity (formulas (1.9) of /7/). Further, a suitable transformation of boundary condition of continuity of tangential stresses is provided by the expression C_n^e in terms of $B_m^e, B_m\ (n - 1 \leqslant m \leqslant n + 1)$ (used below for calculating (3.5) and (3.6)) and reduce the problem to two difference equations of fourth order in B_m^e and B_m (formulas (1.19) and (1.16) of /7/ for $i = 1, \delta_1 = 1, \lambda_1 = \lambda$).

For arbitrary value of V_1 the only nonzero components x_1 of forces acting on the spheres are of the form

$$\mp\pi\mu_e aV_1\,\mathrm{sh}\,\eta_1 4\sqrt{2}\sum_{n=1}^{\infty} n(n+1)C_n^e \tag{3.5}$$

Equating (3.5) to zero we obtain the relation between V_1 and G and by the same token function B. Integral (1.1) for S_{13}, by virtue of symmetry and the Gauss—Ostrogradskii theorem, is equal half of the corresponding integral over the sphere of large radius with the center at the coordinate origin. The behavior of flow, as $|\mathbf{x}| \to \infty$ can be established from (3.1)—(3.4), taking into account the equation of continuity and the fact that forces and their moments are equal zero. Thus, under conditions of free motion, we obtain

$$S_{13} = -\frac{2\pi\mu_e V_1 c^2 \sqrt{2}}{3} \sum_{n=1}^{\infty} (2n+1)\, n\,(n+1)\, C_n^{\varepsilon} \tag{3.6}$$

Table 2

λ	$\varepsilon=1$	0,25	0,1	0,01	10^{-4}
0	0	0	0	0	0
	352	754	885	974	984
0.5	62	247	339	411	420
	414	714	783	823	828
1	97	393	545	670	685
	444	691	725	735	735
2	136	558	787	987	1014
	473	661	650	614	608
5	177	750	1090	1434	1488
	500	614	537	408	383
10	197	853	1268	1749	1843
	511	583	458	235	180
10^7	223	1001	1556	2461	3158
	524	527	312	—220	—744

Table 3

λ	$\varepsilon=1$	0,25	0,1	0,01	10^{-4}
0	9186	7606	6529	4586	2800
	821	2529	3836	6576	9437
0.5	8842	6676	5207	2503	384
	1095	3127	4620	7739	10457
1	8669	6215	4591	1862	226
	1241	3459	5018	7942	9895
2	8493	5743	3978	1330	130
	1393	3825	5457	8137	9492
5	8315	5252	3347	852	62
	1552	4235	5960	8381	9231
10	8233	5021	3049	641	35
	1626	4438	6218	8522	9155
∞	8133	4733	2676	381	4
	1717	4698	6558	8731	9100

The sums of series (3.5) and (3.6) were determined numerically over the limits of recurrent sequencies, as in /7/. For each λ, ε pair a column of quantities $B\cdot 10^4$, $-(K+L)\cdot 10^4$ appears in Table 2. The calculations were carried out on a computer with doubled accuracy, which is essential for the passing to limit, as $\lambda \to \infty$ and small ε; for $\lambda = \infty$ the proposed here method is directly inapplicable /7/. In the case of solid spheres a number of values of η_1 /1/ the function $B(\varepsilon)$ $(\varepsilon = 2(\operatorname{ch}\eta_1 - 1))$ is tabulated in /4/. It proved that for the same η_1 and $\lambda = 10^7$ the calculation of B by the proposed method yields a complete coincidence with /4/ up to $\varepsilon = 0{,}0006$; simultaneously, this method substantially differs from the method of calculation for the solid spheres /1/.

The recurrent formulas /8/ of the method of reflection permit to obtain remote asymptotic representations

$$B = \frac{8\lambda(1+2\lambda)}{(2+3\lambda)(1+\lambda)}\alpha^5 + O(\alpha^8) \tag{3.7}$$
$$K + L = -\frac{2+5\lambda}{2(1+\lambda)}\alpha^3 + \frac{8\lambda}{1+\lambda}\alpha^5 + \frac{(2+5\lambda)^2}{4(1+\lambda)^2}\alpha^6 + O(\alpha^8)$$

where the first term of asymptotics $K+L$ corresponds to results in /9/.

It would be possible to improve formulas (3.7) by supplementary reflections, however the experiment /8/ shows that in this problem when $\lambda \gg 1, \varepsilon \ll 1$ it is difficult to obtain sufficient accuracy owing to slow convergence, hence in the calculation of B and $K+L$ a more universal method /7/ was applied. At the same time by the method of multiple reflections it is possible to prove the unexpected equality $B=0$ when $\lambda = 0$. The theorem of reciprocity /10/ shows that the equality $B=0$ is valid if harmonics p_{-3} are absent in Lamb representation of the velocity field near the spheres during their instantaneous normal motion to the line of centers in the quiescent at infinity medium. The absence of these harmonics when $\lambda=0$ can be established using recurrent formulas /8/ by obtaining relation of the type (2.10).

The analysis of particle trajectories is outside the scope of this paper. We would only point out that integrals of relative motion /4/ and the equality $B=0$ when $\lambda=0$ shows that the region of closed relative trajectories of centers of the two spheres in the steady shear flow in which the conjugate function of distribution is not determined by simple considerations /9/ which for small λ proves to be considerably "finer" than for solid spheres /4/, and vanishes completely for $\lambda = 0$.

4. Solution of problem 3. Using the general solution /11/ of the Stokes equation for the stream function Ψ in coordinates ξ, η, and, also, the symmetry of the problem and the regularity of flows inside the spheres we have

$$\Psi = c^2\sqrt{2}\,(\operatorname{ch}\eta - \mu)^{-3/2} \sum_{n=1}^{\infty} n(n+1)\,\psi_n(\eta)\, Q_n(\mu) \tag{4.1}$$
$$Q_n(\mu) = [P_{n+1}(\mu) - P_{n-1}(\mu)]/(2n+1)$$

$$\psi_n^i = \pm \{A_n \exp[-(n-1/2)|\eta|] + B_n \exp[-(n+3/2)|\eta|]\}$$
$$\psi_n^e = cE_{33}\mathrm{sh}\,\eta \exp[-(n+1/2)|\eta|] + F_n \mathrm{sh}(n-1/2)\eta + H_n \mathrm{sh}(n+3/2)\eta$$

The upper sign corresponds to $i=1$, the lower to the case of $i=2$. The first term of expression for ψ_n^e after substitution in the series /4/ yields the stream function $\Psi_\infty = -1/2\, E_{33}\rho^2 z$ of the unperturbed flow; the additional terms define the velocity perturbation vanishing at infinity.

The first relation of (4.1) has formally the same form as that for motion of drops in medium quiescent at infinity, which permits the boundary conditions to be written using the results in /5,6/ in the form

$$\begin{aligned}&\psi_n^e = \psi_n^1 = R_n,\ d\psi_n^e/d\eta = d\psi_n^1/d\eta \\ &d^2(\psi_n^e - R_n)/d\eta^2 = \lambda d^2(\psi_n^1 - R_n)/d\eta^2,\ \eta = \eta_1 \\ &R_n(\eta) = \frac{V_1^*}{2}\left\{\frac{\exp[-(n-1/2)\eta]}{2n-1} - \frac{\exp[-(n+3/2)\eta]}{2n+3}\right\}\end{aligned} \tag{4.2}$$

According to /11/ the hydrodynamic forces are

$$F_1^z = -F_2^z = -4\pi\mu_e c\sum_{n=1}^{\infty} n(n+1)(F_n + H_n) \tag{4.3}$$

Independent of (4.3) being zero, the integral (1.1) for S_{33} can be calculated using the method as in Sect.3

$$S_{33} = -\frac{16\pi\mu_e c^3}{3}\sum_{n=1}^{\infty} n(n+1)\left[\left(n-\frac{1}{2}\right)F_n + \left(n+\frac{3}{2}\right)H_n\right] \tag{4.4}$$

Taking into account the linearity of the problem and the theorem of reciprocity /10/, we can write

$$\begin{aligned}&F_1^z = -6\pi\mu_e a\,[V_1^*(2\Lambda_{11} - \Lambda_{12}) - lE_{33}D/2] \\ &S_{33} = 4/3\,\pi\mu_e a^3(2+5\lambda)(1+\lambda)^{-1}E_{33}\Gamma - 2\pi\mu_e alV_1^* D\end{aligned} \tag{4.5}$$

where $\Lambda_{11}, \Lambda_{12}$ are the coefficients of resistance /8/ defined by infinite series /5,6/. The explicit expressions can be achieved for coefficients $D(\lambda,\varepsilon), \Gamma(\lambda,\varepsilon)$ in terms of infinite series with the help of (4.1)–(4.4). The respective formulas however, are unwieldy, and are not presented here.

In Table 3 for each pair of λ, ε appears a column of quantities $(1-A)\cdot 10^4$, and $(K - 4/3\,L + 2/3\,M)\cdot 10^4$, obtained using (4.5) and (1.2). Numerical results are supplemented by remote asymptotic representations, obtained by the method of reflections

$$\begin{aligned}&A = \frac{2+5\lambda}{1+\lambda}\alpha^3 - \frac{12\lambda(1+2\lambda)}{(2+3\lambda)(1+\lambda)}\alpha^5 + \frac{(2+5\lambda)^2}{(1+\lambda)^2}\alpha^6 + O(\alpha^7) \\ &K + \frac{4}{3}L + \frac{2}{3}M = \frac{2+5\lambda}{1+\lambda}\alpha^3 - \frac{12\lambda}{1+\lambda}\alpha^5 + \frac{(2+5\lambda)^2}{(1+\lambda)^2}\alpha^6 + O(\alpha^7)\end{aligned} \tag{4.6}$$

The terms of order α^3 correspond to results in /9/. For solid spheres the second term of the asymptotics of A is incorrect in the translation (and does not agree with (4.6) for $\lambda = \infty$), the correct formula appears in the original /4/.

The near asymptotics of functions $A, K + 4/3\,L + 2/3\,M$ can be obtained from (4.5). When $\varepsilon \ll 1$ coefficient Λ_{12} is close to its limit value /6/ for touching spheres. As shown by the analysis of numerical data, the quantity Γ differs from its limit value Γ^* by $O(\varepsilon)$ uniformly over λ, for $\lambda=0$ and $\lambda=\infty$ the coefficient $D(\varepsilon)$ has definitely a finite derivative when $\varepsilon=0$, while for $0<\lambda<\infty$ it differs from this limit value D^* in any case by $o(\sqrt{\varepsilon})$. The quantities Γ^*, D^* were obtained by numerical estimates. Using asymptotics /8/ for Λ_{11}, we have for fixed $\lambda < \infty$

$$\begin{aligned}&\frac{1}{1-A} = \frac{1}{D^*}\left[\frac{\pi^2\lambda}{16\sqrt{\varepsilon}} + \frac{(\lambda^2-3)}{9}\ln\varepsilon + b_0\right] + o(1) \\ &K + \frac{4}{3}L + \frac{2}{3}M = \Gamma^* - 1 - \frac{3D^*(1+\lambda)}{2+5\lambda}(1-A) + O(\varepsilon),\quad \varepsilon \to 0\end{aligned} \tag{4.7}$$

The values of $D^*, b_0, \Gamma^* - 1$ are adduced below:

$\lambda = 0$	0.5	1	2	5	10	∞
$D^* = 1.097$	1.321	1.450	1.596	1.772	1.870	2.039
$b_0 = 0.847$	0.771	0.559	0.021	−0.212	11.009	—
$\Gamma^* - 1 = 1.4041$	1.0963	1.0175	0.9647	0.9303	0.9196	0.9104

It follows from (4.7) that $K + {}^4/_3L + {}^2/_3M$ has an infinite derivative when $\varepsilon = 0$, if $\lambda < \infty$.

The asymptotics for Λ_{11} /8/ and formula (4.7) for $1 - A$ are nonuniformly suitable as $\lambda \to \infty$. In region $\lambda \geqslant 1$, $\varepsilon \ll 1$ a rough but uniformly suitable estimate of $1 - A$ can be obtained using formula /12/

$$\Lambda_{11} \simeq {}^1/_4 f(p)\, \varepsilon^{-1}, \qquad p = 2\lambda \sqrt{\varepsilon}$$

where the functional f is defined in /12/.

When $\lambda < \infty$ and $\varepsilon \to 0$, function $(1 - A)^{-1}$ has an integrable singularity, in consequence of which a coagulation of fluid spheres is possible under the action of macroscopic deformation. However even in strong flows the system of spheres may remain stable owing to the forces of repulsion between the two electric layers of the particle surfaces. Owing to the small action radius these forces can be modelled by contact forces only hindering the coagulation. In such models under the action of strong macroscopic flow there occurs a temporary formation of doublets, and for the determination of mean stress in the system of liquid spheres it is necessary to determine the volume /9/ as well as the surface probability density for the vector separating the centers of the two particles.

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